Question 1

1992 Solns

This opening problem requires some basic analysis of a polynomial graph. Students should have answered Part (a) by considering the sign of the first derivative, noting that it remains negative on bo… sides of the critical point at 0. Part (b) was to be answered by considering the sign of the second derivative. In Part (c), students had to indicate that horizontal tangent lines occur at all three critical points, including the "shelf" point at $x = 0$.

A favorite technique for analyzing the sign of $f'(x)$ and $f''(x)$ is through a "sign chart." Students were required to have labeled such charts clearly, and to indicate what information was being used to draw their conclusions.

1. Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

Solution *Scoring Scale*

Points (See Footnote)

(a) $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$

Sign of f' $\quad + \quad - \quad - \quad +$

$\qquad -1 \quad 0 \quad 1$

Answer: f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$

3 {
1: $f'(x)$
1: Analyzes sign of $f'(x)$ or explicitly sets student's $f'(x) > 0$
1: Answer

(b) $f''(x) = 60x^3 - 30x = 30(2x^2 - 1)$

Sign of f'' $\quad - \quad + \quad - \quad +$

$\qquad -\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}$

How do you know what each sign

Answer: on $\left(-\frac{1}{\sqrt{2}}, 0\right)$

and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$

3 {
1: $f''(x)$
1: Analyzes sign of $f''(x)$ or explicitly sets student's $f''(x) > 0$
1: Answer

(c) $f'(x) = 0$ when $x = -1, 0, 1$

$x = -1 \Rightarrow f(x) = 4; y = 4$

$f(0) = 2; y = 2$

$f(1) = 0; y = 0$

3 {
1: Solves student's $f'(x) = 0$
$< -1 >$ fewer than 3 solutions
1: One answer
1: All other consistent answers (must be at least one)

Note: For "answer only," maximum 2 of 3 if no f'

Question 2

The setup of this problem is a familiar one, beginning with the velocity function of a particle moving along a line and an initial condition on the position of the particle. Instead of asking for the position function, however, Part (a) asks for the "minimum acceleration" of the particle. Since $a(t) = 6t - 12$, the minimum value occurs at the minimum t-value, namely $t = 0$. Since the particle changes direction twice in the given time interval, the "total distance" in Part (b) must be handled carefully, taking the sign changes of $v(t)$ into consideration. Using the position function $x(t)$ that most students will have found earlier, the "average velocity" in Part (c) is $\frac{x(5) - x(0)}{5 - 0} = 4$.

In Part (a), many students attempted to find the minimum acceleration by setting $v'(t) = 0$, forgetting that the problem called for the minimum of $a(t)$, not of $v(t)$. The most common error in Part (b) was to compute the net displacement of the particle, $\int_0^5 v(t)dt$, rather than the total distance traveled, $\int_0^5 |v(t)|dt$. Part (c) does ask for average velocity rather than average speed, and so the distinction between these two concepts is further emphasized.

2. A particle moves along the x-axis so that its velocity at time t, $0 \le t \le 5$, is given by $v(t) = 3(t - 1)(t - 3)$. At time $t = 2$, the position of the particle is $x(2) = 0$.

   (a) Find the minimum acceleration of the particle.

   (b) Find the total distance traveled by the particle.

   (c) Find the average velocity of the particle over the interval $0 \le t \le 5$.

Solution — *Scoring Scale*

Points

(a) $v(t) = 3t^2 - 12t + 9$

$a(t) = 6t - 12$

a is increasing, so a is minimum at $t = 0$.

$a(0) = -12$ is minimum value of a.

3 {
1: Finds $a(t)$
1: Recognizes minimum at $t = 0$
1: Evaluates

(b) Method 1: sign of $v(t)$

+	0	−	0	+

0 — 1 — 3 — 5

$$\int_0^1 v(t)dt - \int_1^3 v(t)dt + \int_3^5 v(t)dt$$

$$= (t^3 - 6t^2 + 9t)\Big|_0^1 - (t^3 - 6t^2 + 9t)\Big|_1^3 + (t^3 - 6t^2 + 9t)\Big|_3^5$$

$$= 4 - (-4) + 20 = 28$$

4 {
2: Integral expression
1: Antidifferentiation
1: Evaluation

Note: If no change of direction when $t = 1$ and $t = 3$, then eligible for only the antidifferentiation point; maximum 1 of 4

Question 3

The first portion of the Calculus AB course outline is devoted to elementary function topics, which tested in this deceptively simple-looking problem. Part (a) requires some familiarity with the natural logarithm and absolute value functions, but can be answered quickly by knowledgeable students. The same is true of Part (b), although the justification must indicate clearly that students know why *this* particular function is even. Part (c) requires students to find a nontrivial derivative, and subsequently to analyze the sign of the resulting rational function to conclude that *f* has a relative maximum at $x = \pm 1$. Part (d) may appear to be routine, but it is quite challenging. To determine the range, students must combine the analysis of the preceding parts with the additional observation that *f* is unbounded below. Although the graph of *f* is never mentioned, savvy students should have realized that a graph will reveal the answers to each of the questions posed in this problem.

3. Let f be the function given by $f(x) = \ln\left|\frac{x}{1+x^2}\right|$.

   (a) Find the domain of f.

   (b) Determine whether f is an even function, an odd function, or neither. Justify your conclusion.

   (c) At what values of x does f have a relative maximum or a relative minimum? For each such x, use the first derivative test to determine whether $f(x)$ is a relative maximum or a relative minimum.

   (d) Find the range of f.

Solution — *Scoring Scale* — Points

(a) $x \neq 0$

1: Answer

(b) Even

$$f(-x) = \ln\left|\frac{-x}{1+(-x)^2}\right| = f(x)$$

2 { 1: Answer
1: Justification using specific $f(x)$

(c)
$$f'(x) = \left(\frac{1+x^2}{x}\right)\left(\frac{(1+x^2)-2x^2}{(1+x^2)^2}\right) = \frac{1-x^2}{x(1+x^2)}$$

sign of f'
$$\begin{array}{ccccccc} + & & - & & + & & - \\ \hline & -1 & & 0 & & 1 & \end{array}$$

at $x = 1$, $f(x)$ has rel max

at $x = -1$, $f(x)$ has rel max

4 { 2: Derivative
1: Finds critical numbers for student's $f'(x)$ (must have at least two nonzero numbers)
1: Sign analysis of $f'(x)$ and conclusion

(d) Max is $f(1) = \ln\frac{1}{2} = -\ln 2$

$$\lim_{x\to 0^+} f(x) = -\infty$$

OR

$$\lim_{x\to 0^-} f(x) = \lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = -\infty$$

$\therefore$ range $= \left(-\infty, \ln\frac{1}{2}\right]$

2: Answer: $\left(-\infty, \ln\frac{1}{2}\right]$

< −1 > Bounded below

< −1 > Incorrect upper bound

< −1 > Incorrect parenthesis

< −2 > Unbounded above

OR

Method 2: $x(t) = t^3 - 6t^2 + 9t - 2$

$[(\text{or})\ x(t) = t^3 - 6t^2 + 9t + C]$

t	x	
0	-2	
		4
1	2	
		4
3	-2	
		20
5	18	
		28

OR

4 {

1: Antiderivative with correct C
[or indefinite C]

3 {

- 1: Evaluates $x(t)$ when $t = 0, 1, 3, 5$
- 1: Evaluates distances between points
- 1: Evaluates total distance

}

}

$< -3 >$ if no change of direction when $t = 1$ and $t = 3$

(c) Method 1: $$\frac{\int_0^5 (3t^2 - 12t + 9)dt}{5 - 0}$$

$$= \frac{1}{5}\left(t^3 - 6t^2 + 9t\right)\Big|_0^5 = \frac{1}{5}(20) = 4$$

OR

Method 2: $$\frac{x(5) - x(0)}{5 - 0} = \frac{18 - (-2)}{5} = 4$$

2 {

- 1: Applies definition of average velocity
- 1: Evaluation

}

Question 4

The derivative in Part (a) of this problem is found by implicit differentiation, and happens to emerge as a function of y.

Parts (b) and (c) then pose two familiar questions involving the derivative, but students were required to cope with the unfamiliar implications of dealing with a derivative in terms of y rather than x. Part (b) requires some trigonometric analysis in order to go between domain and range values of the sine function. Finding the second derivative in Part (c) involves a subtle chain rule implication that many students frequently overlooked.

Note that Part (b) requires students to find values where the derivative fails to exist, rather than the more routine problem of finding where the derivative equals zero.

4. Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \le y \le 2\pi$.

   (a) Find $\frac{dy}{dx}$ in terms of y.

   (b) Write an equation for each vertical tangent to the curve.

   (c) Find $\frac{d^2y}{dx^2}$ in terms of y.

Solution	Points	Scoring Scale
(a) $\frac{dy}{dx} - \sin y \frac{dy}{dx} = 1$ $\frac{dy}{dx}(1 - \sin y) = 1$ $\frac{dy}{dx} = \frac{1}{1 - \sin y}$	3	2: Implicit Differentiation < −1 > If not with respect to x (may recoup in Part (a) only) < −2 > for chain rule error or incorrect differentiation of right-hand side < −1 > $\frac{dy}{dx} + \sin y \frac{dy}{dx} = 1$ 1: Solves for $\frac{dy}{dx}$ < −1 > no factoring required
(b) $\frac{dy}{dx}$ undefined when $\sin y = 1$ $y = \frac{\pi}{2}$ $\frac{\pi}{2} + 0 = x + 1$ $x = \frac{\pi}{2} - 1$	3	2: Finds y where $\frac{dy}{dx}$ does not exist 1: Establishes equation that determines where student's $\frac{dy}{dx}$ does not exist 1: Solves that equation for y [must involve trig function] 1: Uses that y solution to give equation of vertical line < −1 > Student does not deal with undefined $\frac{dy}{dx}$ Note: 1 of 3 if student's $\frac{dy}{dx}$ always exists and student says there are no vertical tangents. Maximum 2 of 3 if student's $\frac{dy}{dx} = \frac{1}{1 - \sin x}$
(c) $\frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{1 - \sin y}\right)}{dx}$ $= \frac{-\left(-\cos y \frac{dy}{dx}\right)}{(1 - \sin y)^2}$ $= \frac{\cos y \left(\frac{1}{1 - \sin y}\right)}{(1 - \sin y)^2}$ $= \frac{\cos y}{(1 - \sin y)^3}$	3	2: Implicit Differentiation < −2 > If $\frac{dy}{dx}$ is not a quotient or product in y < −2 > Any calculus error 1: Substitutes for student's $\frac{dy}{dx}$ after differentiation and solves for $\frac{d^2y}{dx^2}$

Question 5

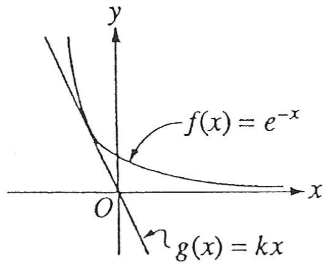
Both differentiation and integration play roles in the solution of this problem. The algebraic conditi on f and f', imposed by having a tangent line that goes through the origin, make it possible to find both the slope of the line and the point of tangency. Once these are determined, finding the area and volume in Parts (b) and (c) is fairly routine.

Many students had difficulty solving for both values called for in Part (a). Students whose algebraic skills failed them in Part (a) were nevertheless eligible for all the points in Parts (b) and (c) if they carried through those parts with parameters that were clearly shown to be negative. Thus, students who were able to understand the geometric conditions in Part (a) could earn all five points in Parts (b) and (c), even if they were unable to answer Part (a).

5. Let f be the function given by $f(x) = e^{-x}$, and let g be the function given by $g(x) = kx$, where k is the nonzero constant such that the graph of f is tangent to the graph of g.
   - (a) Find the x-coordinate of the point of tangency and the value of k.
   - (b) Let R be the region enclosed by the y-axis and the graphs of f and g. Using the results found in part (a), determine the area of R.
   - (c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving the region R, given in part (b), about the x-axis.

Solution *Scoring Scale*

(a)



Points

$$f'(x) = -e^{-x};\ g'(x) = k$$
$$-e^{-x} = k$$
$$e^{-x} = kx$$
$$x = -1 \text{ and } k = -e$$

4
1: Finds $f'(x)$ and $g'(x)$ explicitly
1: Sets student's $f'(x)$ equal to student's $g'(x)$ explicitly
1: Sets $f(x) = g(x)$
1: Solves for x and k
Note: 1 of 4 if student only shows correct answer

(b) $\int_{-1}^{0}\left(e^{-x}-(-ex)\right)dx=\int_{-1}^{0}(e^{-x}+ex)dx$

$$=\left(-e^{-x}+\frac{ex^2}{2}\right)\Bigg]_{-1}^{0}$$

$$=(-1+0)-\left(-e+\frac{e}{2}\right)$$

$$=\frac{e}{2}-1$$

3 {

2: Sets up integral

$< -2 >$ Integrand error including reversal (can be recouped)

$< -1 >$ Limit error(s)

- Missing limit(s)
- Neither limit is 0
- Limit(s) inconsistent with Part (a)

1: Antidifferentiates and evaluates integral

Note: If no value of x is determined in Part (a), then the integral must be of the form $\int_{a}^{0}(e^{-x}-kx)dx$, where a is clearly indicated to represent an unknown negative number.

Note: 0 of 3 if $k=0$ or k is a function of x

(c) $\pi\int_{-1}^{0}\left((e^{-x})^2-(-ex)^2\right)dx$

2 {

2: Sets up integral

$< -2 >$ Incorrect integrand other than $(\text{top})^2-(\text{bottom})^2$ consistent with Part (b)

$< -1 >$ Limit error(s)

- Missing limit(s)
- Neither limit is 0
- Incorrect limit(s) different from Part (b)

$< -1 >$ Error involving π

Note: 0 of 2 if $k=0$

Question 6

The final question on the Calculus AB Examination appears at first to be a related rate problem, but the proportionality condition gives rise to a separable differential equation. A pair of (*t*, *r*) conditions enables students, having solved the differential equation, to find both the constant of proportionality and the constant of integration. Once an expression for *r* in terms of *t* is found, Part (b) requires students to make a simple evaluation.

Some students went directly from the differential equation to an exponential function as the solution. They needed to remember that separable differential equations are now a topic in the Calculus AB syllabus.

6. At time t, $t \ge 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. (The volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

   (a) Find the radius of the sphere as a function of t.

   (b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?

	Solution	Points	*Scoring Scale*
(a)	$\frac{dV}{dt} = \frac{k}{r}$ $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\frac{k}{r} = 4\pi r^2 \frac{dr}{dt}$ $k\,dt = 4\pi r^3 dr$ $\therefore kt + C = \pi r^4$ At $t = 0, r = 1$, so $C = \pi$ At $t = 15, r = 2$, so $15k + \pi = 16\pi, k = \pi$ $\pi r^4 = \pi t + \pi$ $r = \sqrt[4]{t + 1}$	7	1: $\frac{dV}{dt} = \frac{k}{r}$ 1: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ 1: Separates variables 1: Integration (must have C) 1: Solves for C 1: Solves for k 1: Answer (must have solved for 2 constants) Note: If no differential equation solved, then eligible only for first three points
(b)	At $t = 0, r = 1$, so $V(0) = \frac{4}{3}\pi$ $27\,V(0) = 27\left(\frac{4}{3}\pi\right) = 36\pi$ $36\pi = \frac{4}{3}\pi r^3$ $r = 3$ $\sqrt[4]{t + 1} = 3$ $t = 80$	2	1: $r = 3$ $\left[\text{or } 36\pi = \frac{4}{3}\pi\left(\sqrt[4]{t + 1}\right)^3\right]$ 1: Answer Note: If student's $r(t)$ is not student's solution to a differential equation, then eligible only for "$r = 3$" point